CANONICAL MAPS BETWEEN POISSON BRACKETS IN EULERIAN AND LAGRANGIAN DESCRIPTIONS OF CONTINUUM MECHANICS

Darryl D. HOLM [a], Boris A. KUPERSHMIDT [a,b] and C. David LEVERMORE [c]

[a] *Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
[b] *University of Tennessee Space Institute, Tullahoma, TN 37388, USA*
[c] *Lawrence Livermore Laboratory, Livermore, CA 94550, USA*

Received 15 September 1975

Noncanonical, Lie algebraic, hamiltonian strucutures in the eulerian description of ideal continuum mechanics are shown to be compatible with nearly canonical structures in the lagrangrian description. Examples are given for compressible ideal fluid dynamics, magnetohydrodynamics, nonlinear elasticity, multifluid plasmas, superfluids ^{4}He and ^{3}He-A, and chromohydrodynamics.

Introduction. Hamiltonian formulations in terms of canonical Poisson brackets in the eulerian description of continuum mechanics have traditionally been given by the introduction of auxiliary "Clebsch potentials,"some of which are unphysical, but are required in order to complete the hamiltonian structure. For reviews of the traditional eulerian Clebsch method and details of its application, see, e.g., refs. [1–5].

Recently, noncanonical Poisson brackets have also been introduced for various nonlinear field theories, including Maxwell–Vlasov equations [6], magnetohydrodynamics [4,7], multifluid plasma dynamics [4,8], nonlinear elasticity [4,9], superfluids [5,9], and even chromohydrodynamics, which is the nonabelian extension of plasma physics to Yang–Mills fields [10,11]. For a survey of some of the noncanonical brackets, see, e.g., [12]. In each of these cases, noncanonical Poisson brackets are essential to the hamiltonian formulation of the theory in terms of physical variables in the eulerian description.

One approach to finding these noncanonical Poisson brackets in the eulerian physical variables is to show (as a first step) that under a certain map ϕ, a canonical hamiltonian structure in Clebsch variables is compatible with a noncanonical structure, expressed in terms of the physical variables of the theory. The map ϕ relates the physical variables to Clebsch potentials and their gradients.

A difficulty in principle with such noncanonical Poisson brackets, even those compatible with canonical brackets, is to verify that they do, indeed, satisfy the Jacobi identity. However, if the brackets so derived are linear in their variables, then they can be associated readily with appropriate Lie algebras. This guarantees that the Jacobi identity is satisfied. Applications of this approach appear, e.g., in refs. [4,5,10,11,13].

In contrast to the eulerian description, the lagrangian description of continuum mechanics has a nearly *canonical* hamiltonian formulation with physically meaningful (unlike the usual Clebsch representation), canonically conjugate positions and momenta reminiscent of particle dynamics, plus additional fluid variables, whose meaning and Poisson brackets will be discussed below. Below, we simply call the hamiltonian formulation "canonical" in the lagrangian description.

In the present work, we show that the canonical structures in the lagrangian description are compatible with noncanonical hamiltonian structures of continuum mechanics in the eulerian description, when the map ϕ is the natural lagrangian-to-eulerian map which changes both the independent and dependent variables. As far as we are aware, the first discussion of such compatibility of hamiltonian structures in eulerian and lagrangian descriptions

0.031-9163/83/0000–0000/$ 03.00

of hydrodynamics appears (for the one-dimensional case) in the interesting and stimulating paper [14]. More recently, the idea of connecting Poisson brackets for fluids in the lagrangian and eulerian descriptions has reappeared in the case of superfluid ^{4}He [15]. Examples of these compatible hamiltonian structures are given here (in the n-dimensional case) for ideal compressible fluid dynamics, magnetohydrodynamics, nonlinear elasticity, multifluid plasmas, superfluids ^{4}He and ^{3}He-A, and chromohydrodynamics.

Basic set-up for commuting frozen-in variables. In the lagrangian description, a fluid element is labeled by its lagrangian coordinate l_i, $i = 1, \dots, n$. It moves along a trajectory $\boldsymbol{x}(\boldsymbol{l}, t) \in \mathbf{R}^n$ with a certain (canonically conjugate) momentum $\boldsymbol{\pi}(\boldsymbol{l}, t)$. The fluid motion transports mass, entropy, and other so-called "frozen-in" variables of the fluid (e.g., magnetic field). The frozen-in variables $C^{0(\beta)}_{i_1 i_2 \dots i_{k(\beta)}}$, $\beta = 1, 2, \dots, m$, are components of $k(\beta)$-forms (or other tensors). Being frozen-in, they prescribe initial conditions for the flow and, thus, are independent of time.

We use the following notation: $\Lambda^k = \Lambda^k(\mathbf{R}^n)$ k-forms on $\mathbf{R}^n$; $\mathrm{D} = \mathrm{D}(\mathbf{R}^n)$ vector fields on $\mathbf{R}^n$; Y_j elements of D. D acts upon itself by commutation of vector fields and acts upon Λ^{n-k} by Lie derivation, denoted, e.g., $Y(\xi)$ for $\xi \in \Lambda^{n-k}$. The symbol $\ominus$ denotes semidirect product; $\oplus$ denotes direct sum. Latin indices $j = 1, 2, \dots, n$; $\partial_k = \partial/\partial x_k$; $\alpha_k = \partial\alpha/\partial x_k$; and $\delta G/\delta\alpha$ the functional derivation of G with respect to dependent variable α. Lagrangian time derivative is denoted by "dot", e.g., $\dot{\alpha}$; eulerian time derivative, $\alpha_{,t}$. Sum on repeated indices, except where the indices are enclosed in parentheses.

In terms of canonical variables x_i, π_i, and the frozen-in variables $C^{0(\beta)}_{i_1 i_2 \dots i_{k(\beta)}}$, the hamiltonian structure is very simple in the lagrangian description

$$\dot{x}_i = \delta H/\delta\pi_i\,, \quad \dot{\pi}_i = -\,\delta H/\delta x_i\,, \quad \dot{C}^{0(\beta)}_{i_1 i_2 \dots i_{k(\beta)}} = 0\,, \tag{1}$$

where H is the hamiltonian. Thus, for functionals G, H, of $\{x_i, \pi_i, C^{0(\beta)}_{i_1 i_2 \dots i_{k(\beta)}}\}$ one has $\dot{G} = \{H, G\}$ with Poisson bracket

$$\{H, G\} = \int [(\delta H/\delta\pi_i)(\delta G/\delta x_i) - (\delta G/\delta\pi_i)(\delta H/\delta x_i)]\, \mathrm{d}^n l\,, \tag{2}$$

and canonical hamiltonian matrix

$$\mathbf{B} = \begin{array}{c|ccc} & x_j & \pi_j & C^{0(\gamma)}_{j_1 j_2 \dots j_{k(\gamma)}} \\ \hline x_i & 0 & \delta_{ij} & 0 \\ \pi_i & -\delta_{ij} & 0 & 0 \\ C^{0(\beta)}_{i_1 i_2 \dots i_{k(\beta)}} & 0 & 0 & 0 \end{array}\,. \tag{3}$$

In (3), the rows and columns are labeled by the entries in their corresponding Poisson brackets.

As we shall show, the physically meaningful lagrangian-to-eulerian (L–E) map produces a noncanonical, Lie algebraic, hamiltonian structure in the eulerian description which is compatible with the original canonical structure (3). The L–E map ϕ_{LE} from the lagrangian space

$$\mathrm{L} = \{\boldsymbol{l}, t; \boldsymbol{z}\} = \{l_i, t; x_i, \pi, C^{0(\beta)}_{i_1 i_2 \dots i_{k(\beta)}}\}$$

onto the eulerian space

$$\mathrm{E} = \{\boldsymbol{x}, t; \boldsymbol{\upsilon}\} = \{x_i, t; M_i, C^{(\beta)}_{i_1 i_2 \dots i_{k(\beta)}}\}$$

is given by the formulas

$$t = t\,, \quad \mathrm{d}x_i = \dot{x}_i \mathrm{d}t + F_{ij}\mathrm{d}l_j\,, \quad M_i = \pi_i/J\,, \quad J := \det F, \tag{4a}$$

$$C^{(\beta)}_{i_1 i_2 \dots i_{k(\beta)}} = C^{0(\beta)}_{j_1 j_2 \dots j_{k(\beta)}} F^{-1}_{j_1 i_1} F^{-1}_{j_2 i_2} \dots F^{-1}_{j_{k(\beta)} i_{k(\beta)}}\,, \tag{4b}$$

so that $F_{ij} = \partial x_i/\partial l_j$. More explicitly, for some fixed eulerian position $\boldsymbol{y}$, one has

$$C^{(\beta)}_{i_1 i_2 \ldots i_{k(\beta)}}(\boldsymbol{y}) = \int \mathrm{d}^n l \, \delta^n(\boldsymbol{x}(\boldsymbol{l}) - \boldsymbol{y}) \, J C^{0(\beta)}_{j_1 j_2 \ldots j_{k(\beta)}}(\boldsymbol{l}) F^{-1}_{j_1 i_1} F^{-1}_{j_2 i_2} \ldots F^{-1}_{j_{k(\beta)} i_{k(\beta)}} , \tag{4c}$$

where $\delta^n(\boldsymbol{x}(\boldsymbol{l}) - \boldsymbol{y})$ is the n-dimensional Dirac delta function and the integral is taken over the lagrangian domain. Important examples of the L–E map for frozen-in variables are known from magnetohydrodynamics and have motivated the general formulas (4b) and (4c), namely: specific entropy $\eta \in \Lambda^0$ with $\eta = \eta^0$; magnetic vector potential $A \in \Lambda^1$, $A_i = A^0_j F^{-1}_{ji}$; magnetic flux $B = \mathrm{d}A \in \Lambda^2$, $B_{ij} = B^0_{kl} F^{-1}_{ki} F^{-1}_{lj}$; magnetic field $\lambda \in \Lambda^{n-1}$, $\lambda_i = F_{ij}\lambda^0_j/J$; mass density $\rho \in \Lambda^n$, $\rho = \rho^0/J$, and entropy density $\sigma \in \Lambda^n$, $\sigma = \sigma^0/J$.

The hamiltonian matrix $\mathbf{B}$ in (3) tranforms under the L–E map $\phi_{\mathrm{LE}}: \boldsymbol{z}(\boldsymbol{l}) \to \boldsymbol{\upsilon}(\boldsymbol{x}) = \phi[\boldsymbol{z}, \boldsymbol{l}]$, $\boldsymbol{l} \to \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{l})$, according to a general formula,

$$\mathbf{b}^* = (\mathrm{D}\boldsymbol{\upsilon}/\mathrm{D}\boldsymbol{z}) \, \mathbf{B} (\mathrm{d}m_x/\mathrm{d}m_l) (\mathrm{D}\boldsymbol{\upsilon}/\mathrm{D}\boldsymbol{z})^\dagger , \tag{5}$$

where $\mathrm{D}\boldsymbol{\upsilon}/\mathrm{D}\boldsymbol{z}$ is the Fréchet derivative and † denotes adjoint with respect to the measure $\mathrm{d}m_l$. For the L–E map, the ratio of measures $\mathrm{d}m_x/\mathrm{d}m_l = J$, and straightforward calculation of the Fréchet jacobian operator $\mathrm{D}\boldsymbol{\upsilon}/\mathrm{D}\boldsymbol{z}$ followed by matrix multiplication as in (5), gives the general result for the hamiltonian matrix in the eulerian description.

$$-\mathbf{b} = \begin{array}{c|cc} & M_j & C^{(\gamma)}_{j_1 j_2 \ldots j_{k(\gamma)}} \\ \hline M_i & M_j \partial_i + \partial_j M_i & (\mathrm{D}C^{(\gamma)}_{j_1 j_2 \ldots j_{k(\gamma)}}/\mathrm{D}x_i)^\dagger \\ C^{(\beta)}_{i_1 i_2 \ldots i_{k(\beta)}} & -\mathrm{D}C^{(\beta)}_{i_1 i_2 \ldots i_{k(\beta)}}/\mathrm{D}x_j & 0 \end{array} , \tag{6}$$

where, as in (3), the rows and columns are labeled by entries in the corresponding Poisson brackets. The Poisson bracket associated with (6) takes the form

$$\{H, G\} = -\int \mathrm{d}^n x \, \{(\delta G/\delta M_i)[(M_j \partial_i + \partial_j M_i)(\delta H/\delta M_j) + (\mathrm{D}C^{(\gamma)}_{j_1 j_2 \ldots j_{k(\gamma)}}/\mathrm{D}x_i)^\dagger (\delta H/\delta C^{(\gamma)}_{j_1 j_2 \ldots j_{k(\gamma)}})]$$
$$- (\delta G/\delta C^{(\beta)}_{i_1 i_2 \ldots i_{k(\beta)}})(\mathrm{D}C^{(\beta)}_{i_1 i_2 \ldots i_{k(\beta)}}/\mathrm{D}x_j)(\delta H/\delta M_j)\} , \tag{7}$$

where the Fréchet derivative computed from (4b) can readily be shown to equal the "Lie derivative" form,

$$-\mathrm{D}C^{(\beta)}_{i_1 i_2 \ldots i_{k(\beta)}}/\mathrm{D}x_j = C^{(\beta)}_{i_1 i_2 \ldots i_{k(\beta)},j} + C^{(\beta)}_{j i_2 \ldots i_{k(\beta)}} \partial_{i_1} + C^{(\beta)}_{i_1 j \ldots i_{k(\beta)}} \partial_{i_2} + \ldots + C^{(\beta)}_{i_1 i_2 \ldots j} \partial_{i_{k(\beta)}} . \tag{8}$$

Five special cases of physical importance are: Λ^0, Λ^1, Λ^2, Λ^{n-1}, Λ^n. The corresponding negatives of the Fréchet derivatives appearing in the Poisson bracket (7) are, respectively:

$$C^{(0)}_{,j}, \quad C^{(1)}_{i_1,j} + C^{(1)}_j \partial_{i_1}, \quad C^{(2)}_{i_1 i_2,j} + C^{(2)}_{j i_2} \partial_{i_1} + C^{(2)}_{i_1 j} \partial_{i_2}, \quad \partial_j C^{(n-1)}_{i_1} - C^{(n-1)}_m \partial_m \delta_{i_1 j}, \quad \partial_j C^{(n)} .$$

The Poisson bracket (7) is the natural Poisson bracket on the dual to the semidirect product Lie algebra $\mathrm{D} \circledS [\oplus_\beta \Lambda^{n-k(\beta)}]$. The corresponding Lie algebraic commutator is, thus,

$$[(Y; \underset{\beta}{\oplus} \xi^{(\beta)}), (\bar{Y}; \underset{\beta}{\oplus} \bar{\xi}^{(\beta)})] = ([Y, \bar{Y}]; \underset{\beta}{\oplus} (Y(\bar{\xi}^{(\beta)}) - \bar{Y}(\xi^{(\beta)}))) . \tag{9}$$

Dual coordinates are: $\boldsymbol{M}$ dual to $Y \in \mathrm{D}$, and $C^{(\beta)}$ dual to $\xi^{(\beta)} \in \Lambda^{n-k(\beta)}$.

Thus, we naturally recover the *semidirect product* structure in the eulerian description (previously obtained by "experimental computation" in refs. [4,5,10,11,13], from the *direct product* structure in the lagrangian description. The direct sum $\oplus_\beta$ appears in (9) since *different frozen-in variables do not interfere amongst themselves under the L–E map; additional frozen-in variables simply extend the direct sum.*

Remark. If P is a manifold with a Poisson bracket on it and $\alpha : \mathrm{P} \to \mathrm{L}$ is a canonical map, then the composition $\alpha \circ \phi_{\mathrm{LE}} : \mathrm{P} \to \mathrm{E}$ is also canonical. In particular, let $\mathrm{P} = \mathrm{T}*\mathrm{G} \times \mathrm{T}*\mathrm{V}$, where G is the group of diffeomorphisms of $\mathbf{R}^n$ (whose Lie algebra is D), and V * is the space of frozen-in variables. Taking $\alpha = \mathrm{id}_{\mathrm{T}*\mathrm{G}} \times (\mathrm{proj\ on\ V}*)$ (as was suggested to us by Marsden and Weinstein), one obtains the canonical map gotten in ref. [16] by another method.

Applications of the basic set-up for commuting variables. For ideal magnetohydrodynamics (MHD) the physical variables are given by: ρ, mass density; σ, entropy density; $\boldsymbol{M}$, fluid momentum density; and either magnetic vector potential A_i, or magnetic flux $B_{ij} = A_{i,j} - A_{j,i}$. Via relations (4b) for the frozen-in variables $\rho, \sigma \in \Lambda^n$ and $A = A_i \mathrm{d}x_i \in \Lambda^1$, the L–E map takes the canonical bracket (3) into the following Poisson bracket in eulerian physical variables,

$$\{H, G\} = -\int \mathrm{d}^n x\, \{(\delta G/\delta M_i)[(M_j\partial_i + \partial_j M_i)(\delta H/\delta M_j) + \rho\partial_i(\delta H/\delta\rho) + \sigma\partial_i(\delta H/\delta\sigma) + (-A_{j,i} + \partial_j A_i)(\delta H/\delta A_j)]$$
$$+ (\delta G/\delta\rho)\,\partial_j\rho(\delta H/\delta M_j) + (\delta G/\delta\sigma)\,\partial_j\sigma(\delta H/\delta M_j) + (\delta G/\delta A_i)(A_{i,j} + A_j\partial_i)(\delta H/\delta M_j)\}\,. \qquad (10)$$

In obvious notation, (10) is defined to be the sum,

$$\{H, G\} =: \{H, G\}_M + \{H, G\}_\rho + \{H, G\}_\sigma + \{H, G\}_A\,.$$

Thus, one recovers the Poisson bracket for MHD found in ref. [4] by another method and identified there as associated to the semidirect product $\mathrm{D} \circledS (\Lambda^0 \oplus \Lambda^0 \oplus \Lambda^{n-1})$. In the same way, but in terms of different physical variables $\{\rho, \sigma, M_i, B_{ij}\}$, with $B = B_{ij}\mathrm{d}x_i \wedge \mathrm{d}x_j \in \Lambda^2$, one finds the Poisson bracket,

$$\{H, G\} = -\int \mathrm{d}^n x\, \{(\delta G/\delta M_i)[(M_j\partial_i + \partial_j M_i)(\delta H/\delta M_j) + \rho\partial_i(\delta H/\delta\rho) + \sigma\partial_i(\delta H/\delta\sigma)$$
$$+ (-B_{jk,i} + \partial_j B_{ik} + \partial_k B_{ji})(\delta H/\delta B_{jk})] + (\delta G/\delta\rho)\,\partial_j\rho(\delta H/\delta M_j) + (\delta G/\delta\sigma)\,\partial_j\sigma(\delta H/\delta M_j)$$
$$+ (\delta G/\delta B_{mn})(B_{mn,j} + B_{jn}\partial_m + B_{mj}\partial_n)(\delta H/\delta M_j)\}\,,$$
$$\{H, G\} = \{H, G\}_M + \{H, G\}_\rho + \{H, G\}_\sigma + \{H, G\}_B\,. \qquad (11)$$

This Poisson bracket is also discussed in ref. [4] and is identified there as living on the dual to the Lie algebra $\mathrm{D} \circledS (\Lambda^0 \oplus \Lambda^0 \oplus \Lambda^{n-2})$. Either one of the Poisson brackets (10) or (11) generates the equations of motion for MHD as a hamiltonian system $\dot{G} = \{H, G\}$ with hamiltonian

$$H = \int \mathrm{d}^n x\, [|\boldsymbol{M}|^2/2\rho + \rho e(\rho, \sigma/\rho) - \tfrac{1}{4}\mathrm{Tr}\,\mathbf{B}^2]\,, \qquad (12)$$

where $\mathrm{Tr}\,\mathbf{B}^2 = (A_{i,j} - A_{j,i})(A_{j,i} - A_{i,j})$. When the magnetic fields are absent, the hamiltonian, equations of motion, and Poisson brackets all reduce to those for compressible (adiabatic) fluid dynamics.

Next, in ideal elasticity, the physical variables include those for adiabatic fluid dynamics, $\rho, \sigma, \boldsymbol{M}$, as well as the frozen-in lagrangian displacements $\theta_{(k)} \in \Lambda^0(k)$, $k = 1, 2, \ldots, n$. Via relations (4) the Poisson bracket in eulerian physical variables is given by

$$\{H, G\} = -\int \mathrm{d}^n x \left\{\frac{\delta G}{\delta M_i}\left[(M_j\partial_i + \partial_j M_i)(\delta H/\delta M_j) + \rho\partial_i(\delta H/\delta\rho) + \sigma\partial_i(\delta H/\delta\sigma) - \sum_{k=1}^{n} \theta_{(k),i}(\delta H/\delta\theta_{(k)})\right]\right.$$
$$\left. + (\delta G/\delta\rho)\,\partial_j\rho(\delta H/\delta M_j) + (\delta G/\delta\sigma)\,\partial_j\sigma(\delta H/\delta M_j) + \sum_{k=1}^{n} (\delta G/\delta\theta_{(k)})\,\theta_{(k),j}(\delta H/\delta M_j)\right\},$$

$$\{H, G\} = \{H, G\}_M + \{H, G\}_\rho + \{H, G\}_\sigma + \sum_{k=1}^{n} \{H, G\}_{\theta(k)}\,. \qquad (13)$$

As discussed in [4], the Lie algebra responsible for this bracket is

$$\mathrm{D} \circledS \left(\Lambda^0 \oplus \Lambda^0 \bigoplus_{k=1}^{n} \Lambda^n(k)\right).$$

Hamilton's equations for elasticity are treated, e.g., in ref. [3].

For a third example, in ideal plasma dynamics in the fluid approximation (see, e.g., refs. [4], [8]), the physical variables are: mass density ρ; entropy density σ; self-consistent electric field $\boldsymbol{E}$ and magnetic vector potential $\boldsymbol{A}$; and the total momentum density $\boldsymbol{\pi} = \rho^0(\boldsymbol{u} + (q/m)\boldsymbol{A})$, where $\boldsymbol{u} = \dot{\boldsymbol{x}}$ is particle velocity and q/m is the charge to mass ration of the particles. The equations of motion are simply Hamilton's canonical equations (1), with hamiltonian

$$H = \int \mathrm{d}^n l \rho^0(l)[\tfrac{1}{2}|\boldsymbol{\pi}/\rho^0 - (q/m)\boldsymbol{A}(x,t)|^2 + (q/m)\Phi(x,t) + e(\rho^0/J, \sigma^0/J)] + \int \mathrm{d}^n x(\tfrac{1}{2}|\boldsymbol{E}|^2 - \tfrac{1}{4}\mathrm{Tr}\,\mathbf{B}^2 + \boldsymbol{E}\cdot\boldsymbol{\nabla}\Phi) \tag{14}$$

in a mixed description, which is lagrangian for fluid variables $\boldsymbol{x}, \boldsymbol{\pi}$, and eulerian for the canonically conjugate electromagnetic fields $\boldsymbol{E}, \boldsymbol{A}$.

Mapping the canonical fluid variables $\boldsymbol{x}, \boldsymbol{\pi}$, and the frozen-in variables ρ^0 and σ^0 via (4a), (4c) leads to the following Poisson bracket

$$\{H, G\} = \{H, G\}_M + \{H, G\}_\rho + \{H, G\}_\sigma + \{H, G\}_{E-A}\,, \tag{15}$$

where $\{H, G\}_{E-A}$ is the canonical bracket in $\boldsymbol{E}$ and $\boldsymbol{A}$. For the ideal plasma equations with multiple particle species, the sum over species appears in the lagrangian fluid part of H in (14), and the first three terms in the eulerian bracket (15) acquire summation over species; otherwise, the Poisson structure is unchanged. The resulting multifluid plasma bracket, thus, recovers the form in ref. [4] which is associated to the dual of the direct sum

$$\bigoplus_{s=1}^{N} [\mathrm{D}(s) \ominus (\Lambda^0(s) \oplus \Lambda^0(s))]\,,$$

with species label $s = 1, 2, \ldots, N$, and the following dual coordinates: $\boldsymbol{M}^{(s)} = \rho^{(s)}(\boldsymbol{u}^{(s)} + q^{(s)}/m^{(s)}\boldsymbol{A})$ dual to vector fields in $\mathrm{D}(s)$ and $\rho^{(s)}$, $\sigma^{(s)}$, each dual to functions in $\Lambda^0(s)$.

More General Cases. In practice, most fluid dynamical systems are less elementary than those considered thus far: e.g., superfluids and nonabelian fluids. For such systems, two new features appear in the noncanonical Poisson brackets. First, the hamiltonian matrix associated with the Poisson bracket can contain constant terms, in addition to the linear terms discussed previously. For example, this occurs in the case of superfluid ^{4}He, see, e.g., ref. [4]. Second, some parts of the Poisson bracket may be associated to finite-dimensional Lie algebras, as occurs, e.g., for chromohydrodynamics [10,11] and superfluid ^{3}He-A [4]. When either (or both) of these new features appear, the L–E map continues to provide natural canonical representations, when appropriate changes are made in the previously canonical structure in the lagrangian picture, as follows.

Let us express the Poisson bracket (7) in shorthand notation as

$$\{H, G\} = \{H, G\}_M + \sum_\beta \{H, G\}_{C(\beta)}\,. \tag{16}$$

In the first case, when constant terms appear in the hamiltonian matrix, the bracket (16) becomes

$$\{H, G\} = \{H, G\}_M + \sum_\beta \{H, G\}_{C(\beta)} + \sum_\gamma \{H, G\}_{S(\gamma)}\,, \tag{17}$$

where the subscript $S(\gamma)$ on the last term refers to "symplectic" and

$$\{H, G\}_{S(\gamma)} = -\int \mathrm{d}^n x[(\delta G/\delta C^{(\gamma)})\cdot(\delta H/\delta \widetilde{C}^{(\gamma)}) - (\delta H/\delta C^{(\gamma)})\cdot(\delta G/\delta \widetilde{C}^{(\gamma)})]\,. \tag{18}$$

(In place of "dot" in formula (18) there could stand certain linear differential operators, see ref. [9], but we are not treating this case in the present paper.) The index γ ranges over a subrange of that for β, while $\widetilde{C}^{(\gamma)} \in \Lambda^{n-k(\gamma)}$ for $C^{(\gamma)} \in \Lambda^{k(\gamma)}$. Vector notation is used in (18) for variational derivatives, summed over components of $C^{(\gamma)}$ and $\widetilde{C}^{(\gamma)}$.

In view of the L–E map, the additional terms (18) in bracket (17) arise from lagrangian equations of the form

$$\dot{x}_i = \delta H/\delta \pi_i, \quad \dot{\pi}_i = -\delta H/\delta x_i, \quad \dot{C}^{0(\gamma)} = -\delta H/\delta \widetilde{C}^{0(\gamma)}, \quad \dot{\widetilde{C}}^{0(\gamma)} = \delta H/\delta C^{0(\gamma)} . \tag{19}$$

For example, in the case of superfluid ^{4}He the corresponding Poisson bracket to (17) is (see formula (5) of ref. [5])

$$\{H, G\} = \{H, G\}_M + \{H, G\}_\rho + \{H, G\}_\sigma + \{H, G\}_{S(0)} , \tag{20}$$

where $\boldsymbol{M}$ is total (normal plus superfluid) momentum density, ρ is mass density, σ entropy density (of normal fluid only) and $\{H, G\}_{S(0)}$ involves mass density ρ and superfluid phase α, according to

$$\{H, G\} = -\int \mathrm{d}^n x \, [(\delta G/\delta \rho)(\delta H/\delta \alpha) - (\delta H/\delta \rho)(\delta G/\delta \alpha)] . \tag{21}$$

Therefore, the lagrangian description of superfluid ^{4}He can be gotten from (19) with $C^0 = \rho^0$, $\widetilde{C}^0 = \alpha^0$ and H as a pullback of the eulerian hamiltonian taken from, e.g., ref. [17].

The second additional feature is noncommutativity, which means in the simplest circumstance that some of the $C^{0(\gamma)}$ can take values in the duals, $\mathcal{G}(\gamma)^*$, of various finite dimensional Lie algebras, $\mathcal{G}(\gamma)$. The simplest nonabelian case occurs when $C^{0(\gamma)} \in \Lambda^n \otimes \mathcal{G}(\gamma)^* \approx [\Lambda^0 \otimes \mathcal{G}(\gamma)]^*$, where $\Lambda^0 \otimes \mathcal{G}(\gamma)$ stands for functions on $\mathbf{R}^n$ with values in $\mathcal{G}(\gamma)$. The corresponding additional pieces for the Poisson bracket (7) look as follows,

$$\sum_\gamma \{H, G\}_{(\gamma)} = \sum_\gamma \int \mathrm{d}^n x \, [(\delta G/\delta \rho_a^{(\gamma)})(\delta H/\delta \rho_b^{(\gamma)}) \, \epsilon_{ab}^c(\gamma) \, \rho_c^{(\gamma)}] , \tag{22}$$

where $\epsilon_{ab}^c(\gamma)$ are structure constants of the Lie algebra $\mathcal{G}(\gamma)$ and $\rho_a^{(\gamma)}$, etc., are coordinates on $\Lambda^n \otimes \mathcal{G}(\gamma)^*$. The additional pieces (22) correspond to $\mathrm{D} \ominus [\oplus_\gamma (\Lambda^0(\gamma) \otimes \mathcal{G}(\gamma)]$. Examples where such noncommutative Poisson brackets appear are given in ref. [10], formular (9), and ref. [11], formula (102). In the lagrangian description for such cases, the equations of motion take the form

$$\dot{x}_i = \delta H/\delta \pi_i, \quad \dot{\pi}_i = -\delta H/\delta x_i, \quad \dot{\rho}_a^{0(\gamma)} = (\delta H/\delta \rho_b^{0(\gamma)}) \, \epsilon_{ab}^c(\gamma) \rho_c^{0(\gamma)} , \tag{23}$$

with H given as a pullback of the eulerian hamiltonian in refs. [10,11].

More generally, suppose that the set of all frozen-in variables (not only elements of Λ^0) form a finite-dimensional Lie algebra. Then, after the L–E map, exactly the same Lie algebra will reappear in the Poisson bracket for eulerian variables. This applied, for example, in the case of superfluid ^{3}He-A, see formular (24) of ref. [4].

In conclusion, we point out that the general procedure outlined in this paper can also be used to find noncanonical Poisson brackets for a classical Yang–Mills/Vlasov plasma, described in terms of self-consistent, nonabelian Yang–Mills fields and a distribution function on single-particle phase space, as already done in ref. [11].

This work was supported in part by NSF and DOE. We would like to acknowledge helpful conversations with D. McLaughlin. We have also had useful conversations and correspondence with J. Marsden and A. Weinstein, who (following the ideas of Arnol'd [18]) have independently worked on some of the questions addressed in the first two sections of this paper.

References

[1] J. Serrin, Mathematical principles of classical fluid mechanics, in: Handbuch der Physik VIII/1, Fluid Dynamics I, ed. C. Truesdell (Springer, Berlin, 1959) pp. 125–263.
[2] F.P. Bretherton, J. Fluid Mech. 44 (1970) 19.
[3] R.L. Seliger and G.B. Whitham, Proc. R. Soc. A305 (1968) 1.
[4] D.D. Holm and B.A. Kupershmidt, Physica 6D (1983) 347.
[5] D.D. Holm and B.A. Kupershmidt, Phys. Lett. 91A (1982) 425.
[6] J.E. Marsden and A. Weinstein, Physica 4D (1982) 394.
[7] P.J. Morrison and J.M. Greene, Phys. Rev. Lett. 45 (1980) 790; 48 (1982) 569 (E).
[8] R. Spencer, in: Methods in hydrodynamics and integrability in dynamical systems (La Jolla, 1981), Conf. Proc., eds. M. Tabor and Y. Treve (American Institute of Physics, New York, 1982) pp. 121–126.
[9] I.E. Dzyaloshinskii and G.E. Volovik, Ann. Phys. (N.Y.) 125 (1980) 67.

[10] J. Gibbons, D.D. Holm and B.A. Kupershmidt, Phys. Lett 90A (1982) 281.
[11] J. Gibbons, D.D. Holm and B.A. Kupershmidt, Physica 6D (1983) 179.
[12] M. Tabor and Y. Treve, eds., Methods in hydrodynamics and integrability in dynamical systems (La Jolla, 1981), Conf. Proc. (American Institute of Physics, New York, 1982).
[13] D.D. Holm and B.A. Kupershmidt, Phys. Lett. 93A (1983) 177.
[14] L.J.F. Broer and J.A. Kobussen, Appl. Sci. Res. 29 (1974) 419.
[15] G.E. Volovik and V.S. Dotsenko Jr., JETP Lett. 29 (1979) 576.
[16] J.E. Marsden, T. Ratiu and A. Weinstein, Semidirect products and reduction in mechanics, preprint.
[17] I.M. Khalatnikov and V.V. Lebedev, J. Low Temp. Phys. 32 (1978) 789.
[18] V.I. Arnol'd, Ann. Inst. Fourier (Grenoble) 16 (1966) 319.